Supplementary Document

Setagaya Elementary Mathematics Research Open House
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1 Introduction

From various surveys, we know that students have difficulty writing appropriate calculation expression for multiplication and division involving decimal numbers and fractions. The following data come from the surveys conducted by the Elementary Mathematics Education Research Group of Tokyo Prefecture (EMERG-T).

The EMERG-T has been conducting biennual survey to research students' understanding in the domain of Numbers and Calculations. In the past 5 administrations of the survey, they have used the same problems for Grade 5 multiplication and division of decimal numbers.

(1) Calculation Problems (Calculate the following and write the answer in the box provided.)

i 0.6 × 3.8

Year	2004	2006	2008	2010	2012
Success Ratio (%)	79	80	78	76	77

ii 8 ÷ 1.6

Year	2004	2006	2008	2010	2012
Success Ratio (%)	73	74	73	74	77

(2) Setting up calculation expression (Read the following problem and write the calculation expression to find the answer.)

i There is a 1 meter iron bar that weighs 1.3 kg. How much will a 0.8 meter of the same iron bar weigh?

Year	2004	2006	2008	2010	2012
Success Ratio (%)	57	56	59	59	61

ii With 2.8 L of paint, we can paint 3.5 m^2 of board. How many L of paint do you need to paint 1 m^2 of board?

Year	2004	2006	2008	2010	2012
Success Ratio (%)	46	53	52	54	53

From these data, we can say that children can generally do calculations of "× decimal number" and "÷ decimal number," but they have more difficulty determining the necessary calculations. In reality, this difficulty has been known for some time, and the question of how to improve students' ability to determine the correct calculation involving "× decimal number" and "÷ decimal number" has been a long standing issue in mathematics education.

II Expanding the meaning of multiplication and divison

As stated above, the data from the survey clearly indicates that students can perform calculations but they cannot determine which operation is appropriate. I believe that students' lack of understanding of the meaning of multiplying and dividing by decimal numbers is the source of this difficulty. In other words, students do not understand the expansion of the meaning of multiplication and division.

I believe one of the reasons for this situation is the way how students are taught to determine the appropriate operation. It appears that students are taught to determine the appropriate operation using equations with words.

When we are working with whole numbers, we can consider multiplication as repeated addition. However, when multiplying by decimal numbers or fractions, we cannot use the repeated multiplication interpretation.

For example, consider this problem: One meter of tape costs 80 yen. How much will 2.4 meters of the same tape cost? Some students can determine the answer using the reasoning like this: First, 2 meters will cost 160 yen. Since 0.1 meter of the tape will be 8 yen, 0.4 meter will const 32 yen. Thus, the total costs is 192 yen.

However, it is difficult to explain why this problem will be solved by using the calculation, 80 × 2.4.[1] We can't really say 2.4 sets of 80 yen. This is the reason we have to expand the meaning of multiplication.

However, in many classrooms today, the expression, 80 × 2.4 is justified by the equation with words, (unit price) × (length) = (price).

Even in textbooks, they expect the lesson to flow like this:

T: What would be the calculation if the length of tape was 2 meters?
C: It will be 80 × 2.
T: What if the tape was 3 meters?
C: 80 × 3.
T: The equation with words that tells us how to find the price will be:
[cost for 1 meter] × [length] = [price]
T: What do you think the calculation should be if the length of tape is 2.4 meters?

Textbooks often include a double number line diagram, too, but there was no suggestion for having students engage in interpreting double number lines. A balloon contains a suggestion, "Let's use the equation with words shown above." Then, students will jump into "Let's think about ways to calculate."

In this flow of lesson, we can say that the justification for the operation is based on the equation with the words. Based on this experience, it will be difficult for students to observe proportional relationship from a double number line diagram even if it is presented. Moreover, students will be satisfied with using equation with words to determine the appropriate operation, and they may not feel the need to interpret the double number line. In reality, I suspect there are not many teachers who teach the meaning of double number line carefully. Because students have yet to learn to view situations from a proportional perspective, it will be difficult for them to understand the true meaning of the diagram.

In that case, it is not surprising if students did not understand the meaning of multiplying by 2.4.

Some may argue that we can use the phrase, "2.4 times as much" to explain the meaning of multiplying by 2.4. It is true that the idea of "times as much" is used not only with whole numbers but also with decimal numbers, such as "1.5 times as much," or "2.4 times as much."

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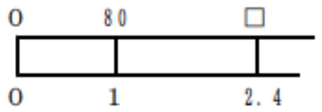
[1] In Japan, the multiplicand (group size) is written first in a multiplication expression. Thus, 3 × 4, for example, means 4 sets of 3, not 3 sets of 4.

But, is it sufficient to conclude "because we are determining 2.4 times as much, we can use × 2.4" based on the wording? I believe it is necessary to explain its meaning based on the meaning of times as much with decimal numbers.

Therefore, it is important for students to re-grasp the meaning of multiplication as, "(base quantity) × (scale factor) = (scaled quantity)[2]." In this way students can expand the meaning of multiplication.

Double number line diagrams are useful tools for supporting students to make sense of the expansion of the meaning of multiplication. For the problem above, the diagram will look like this:



From this diagram, we derive the operation to determine the quantity (□)corresponding to 2.4 (scale factor) as 80 × 2.4. It is essential that students read off this diagram the relationship among the quantities represented in it.

If we say $a \times b = c$ as (base quantity) × (scale factor) = (scaled quantity), the meaning of division is as follows: (1) operation to determine the scale factor (quotitive division), and (2) operation to determine the base quantity (partitive division).

In the case of "operation to determine the scale factor," that is, $c \div a = b$, the quotient (b) tells us the number of groups. However, if b is a decimal number, we interpret the quotient with ties as much. For example, suppose we have the following problem: The height of the library building is 12 meters. The height of my house is 5 meters. How many times as high is the library building as my house?

The equation for this problem is 12 ÷ 5 = 2.4. We interpret the quotient, 2.4, as "If 5 is considered as 1, the number corresponding to 2.4 is 12." This is the meaning of times as much with decimal numbers.

If a in $c \div a = b$ is a decimal number, we interpret its meaning as the operation to answer the question, "If we consider a as 1, what is the scale factor (b) corresponding to c?" Then, we can apply this interpretation not only in the case when they are whole numbers but also when they are decimal numbers, or even fractions.

In the case of "operation to determine the base quantity," it is partitive division if the numbers are whole numbers. However, if b is a decimal number or a fraction, as the survey by the EMERG-T indicates, students have difficulty chossing the appropriate operation and answer the question correctly.

If b is a decimal number or a fraction, we must deal with the challenge of interpreting "equal partitioning." When we partition a quantity into 3 equal groups, we are asked to find the size of one group. Thus, we are calculating "amount per 1." Therefore, we can interpret $c \div b = a$, (scaled quantity) ÷ (scale factor) = (base quantity), as the operation to answer the question, "If b and c correspond to each other, what corresponds to 1?" Then, this interpretation can be applied even when b and c becomes other than whole numbers, i.e., decimal numbers or fractions.

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[2] The Japanese word translated as "scale factor" or "scaled" is *wariai* (割合). It is also translated as "ratio" or "proportion."

III To overcome the issue

As we think about how to overcome this long-standing issue, I believe it must go beyond the instruction of decimal multiplication in Grade 5.

I believe we need to improve the mathematics curriculum, and we want to make two proposals.

(1) We need to include experiences in lower grades that will become foundation for the concept development of *wariai* (scale factor) - using the 1958 National Course of Study (COS).

(2) Starting in Grade 3, as they study multiplication and division, they must learn to represent situations in double number lines and interpreting them. At the same time, we need to provide opportunities for students to utilize double number lines to solve problems. In that way, we help students to learn to use double number lines as reasoning tool.

1 Activities to nurture proportional perspective

I believe it is effective to set up experiences like below in a curriculum to nurture proportional perspective.

[Lower Grades]

The concept of *wariai* (scale factor) is derived from comparing two numbers and quantities. There are many places where students compare numbers and quantities in lower grades. Those situations can serve as foundational experiences. We want to emphasize experiences such as these:

<Grade 1>

1. In a unit on subtraction, students are often asked to find the difference, for example, "How many more blue flowers are there than red flowers?" In those situations, we should help students attend to which quantity is the base of comparison and which quantity is being compared.
2. Students engage in skip counting activities such as counting marbles by 2's and by 5's. Help students understand that we are considering a set of 2 (or 5) marbles as one group (base quantity), and when there are 2 groups, we have 4 (or 10) marbles, with 3 groups 6 (or 15) marbles, with 4 groups 8 (or 20) marbles, and so on.
3. While adding multiples of 10, for example 60 + 30, use manipulatives and help students to understand we have 6 tens and 3 tens. Thus since we have 6 + 3 = 9 tens, the sum is 90. In this way, if we consider 10 as 1, 60 can be considered as 6 and 30 as 3.
4. While comparing lengths, areas, and volumes (capacity), quantify using arbitrary units. In other words, students will use your own unit and engage in measuring activities. Help students become consious that they are considering their unit as 1. For example, if we consider the length of an eraser as "1," then the vertical length of a notebook can be shown with the number "5."

<Grade 2>

The 1958 COS includes a Grade 2 standard, "Help students understand the basic ideas for *wariai* (scale factor) in relation to the study of multiplication and measurement." Furthermore, it also includes statements like the following: "Through the manipulation of concrete materials, help students understand the basic ideas for *wariai* (scale factor). Students will learn expressions like 'twice as much as ~' or '1/3 of ~.'" Based on these ideas, we want Grade 2 students to engage in the following activities.

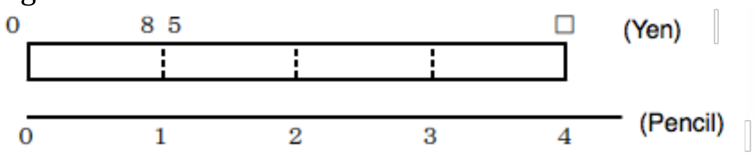
1. In Grade 2, students learn about measuring length and volume (capacity). They will estimate the length or the volume of objects in their surroundings and measure using arbitrary units. These experiences are the foundation for the study of *wariai* (scale factor).
2. The major emphasis in Grade 2 is the study of multiplication. Students first learn multiplication as repeated addition. In the study of multiplication, what is important for students to understand what number and quantity is being grasped as the amount per 1. Therefore, instead of always providing students the per 1 amount, they need to be given opportunities to chose their own per 1 amount and make use of multiplication to find the total amount. Such experiences will be a foundation for the proportional perspective.

[Intermediate Grades]
<Grade 3>

In the 1958 COS, there is the following Grade 3 standard: "Students will gradually extend their proportional perspective and be able to express relationships between quantities using mathematical expressions in simple cases." From the perspective of "gradually extending the proportional perspective," I believe we need to keep in mind the following as we teach.

1. Teaching of multiplication

   In the current COS, teaching of multiplication centeres around learning how to calculate. I believe it is important even in Grade 3 to further deepen students' understanding of the meaning of multiplication. In order to do so, students can represent problem situations in a diagram like the one shown below and grasp the relationship between the two quantities as a proportional relationship. For example, if we have the problem, "We bought 4 pencils. If each pencil costs 85 yen, how much is the total cost?" the diagram will look like this:



   Teach carefully how to interpret this diagram. Students need to grasp the meaning of the correspondence between 85 yen on the tape and 1 on the number line. This correspondence shows that one pencil costs 85 yen. Moreover, they need to understand the correspondence between ☐ and 4 represents the cost (☐) when the number of pencil is 4 times as many.

2. Teaching of division

   Students learn the meaning of division in Grade 3. There are 2 types of division. The first is partitive division. For example, division is the operation to find ☐ in the expression, ☐ × 3 = 24, for the problem situation, "If you share 24 items among 3 people equally, how many does one person receive?" If we represent the problem situation on a double number line, it will look like this:

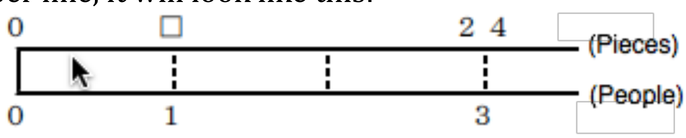


Figure H

The other is quotitive division. For example, division is the operation to find ☐ in the expression, 3 × ☐= 24, for the problem situation, "If you share 24 items by giving each person 3 pieces, with how many people can you share them?" If we represent the problem situation on a double number line, it will look like this:

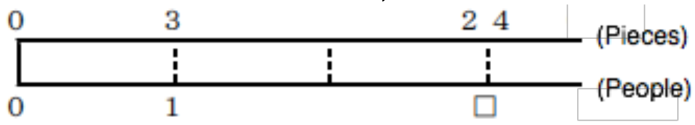


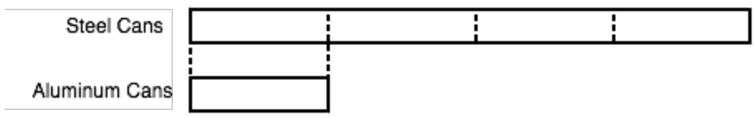
Figure I

It is probably too much for students who are learning division for the first time to draw these diagrams on their own. However, I believe by showing these diagrams as students make sense of the meaning of division, we can lay the groundwork for our issue of "× decimal numbers/fraction" and "× decimal numbers/fractions." Of course, it is essential that we explain carefully how to interpret the diagram so that students understand what it represents.

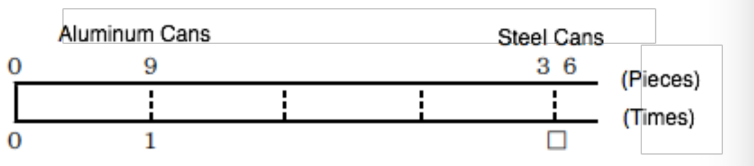
For example, in Figure H, help students read off the diagram that 24 pieces are for 3 people, and ☐ represents the number of pieces for one of 3 equal partitions. Moreover, we would like students to think multiplicatively and notice the inverse relationship - 3 times as much of one person's share is 24 pieces. Then, students can generate the equation, ☐ × 3 = 24, from the diagram. In order to connect to "÷ decimal numbers/fractions," we want to touch upon the viewpoint that ☐ can be considered as the amount per 1.

In the study of "calculation to determine how many times as much," students learn that division is used as the operation. Students are comparing two quantities from the viewpoint of "times as much with whole numbers," however, they are indeed studying *wariai* (scale factor) itself.

In a textbook, they pose the following problem: "We picked up 36 steel cans and 9 aluminum cans. How many times as many steel cans as aluminum cans did we pick? The textbook also includes the following diagram and ask students to determine what calculation is needed.



I would also like to include the double number line shown below as well.



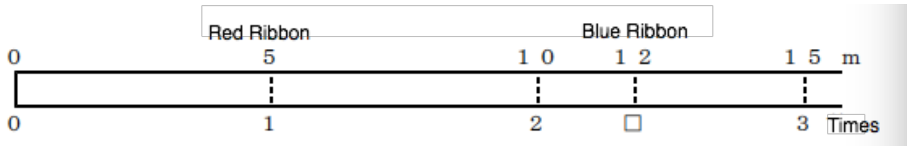
<Grade 4>

One of the 1958 COS standard states, "Students will understand how to represente *wariai* (scale factor) using fractions, and they will represent and interpret mathematical expressions representing relationships between quantities." In addition, as specific standards in the domain of Quantitative Relationships, the COS includes "to deepen students' understanding of *wariai* between two quantities" and "to understand calculations involving *wariai* in simple cases." Based on these standards, we want to focus on the calculation, "Whole Number ÷ Whole Number = Decimal Number." This topic is currently discussed in Grade 5, often times through problem situations such as the following.

The table below summarizes the lengths of various ribbons we have. If we consider the length of the red ribbon as the base, how many times as much are the lengths of other ribbons?

Color	Length (cm)
Red	5
White	10
Blue	12
Yellow	4

First, help students realize that we can determine that the white ribbon is twice as long as the red ribbon by the calculation 10 ÷ 5. Then, we will have students think about what calculation will help us determine how many times as long is the blue ribbon as the red one. What is important here is for students to be able to describe this relationship as "If we consider the length of the red ribbon as 1, what corresponds to 2.5 is the length of the blue ribbon." Even in this situation, we would like to include a double number line such as shown below.



Based on the activity to interpret this diagram, we want students to understand that "to consider the length of the red ribbon as the base quantity" means to consider the length of the red ribbon "as 1," the length of the white ribbon, 10 cm, is the amount corresponding to 2 (that is, 2 times as much), and the quantity, ☐, can be calculated with 12 ÷ 5 and it is 2.4.

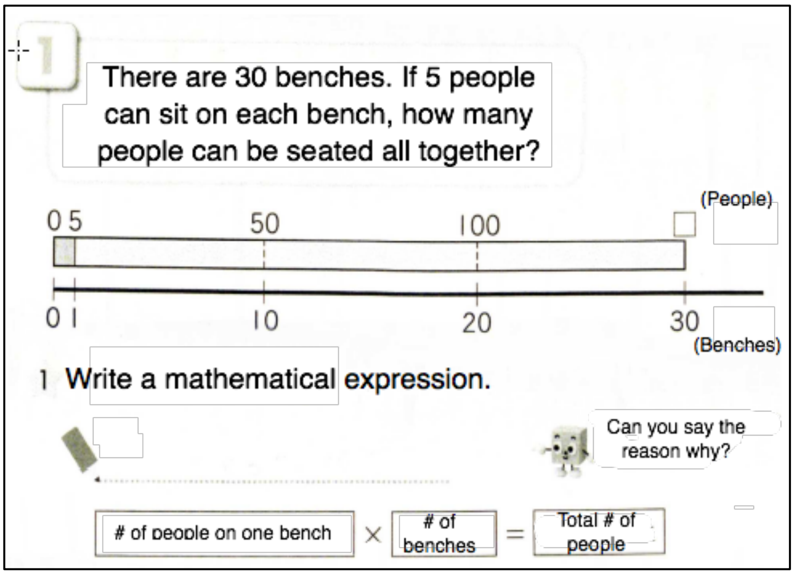
2 Use of double number line

Double number lines are useful tool to grasp the relationship of quantities in a problem situations and determin appropriate operation (multiplication or division) to find the missing quantity. They can also be useful in thinking about ways of calculation.

Although the use of doulbe number lines in teaching mathematics has spread somewhat, I believe it is still limited. I sometime hear the argument that since students can use equations with words to determine the appropriate operation, there is no need to use double number lines which may be very difficult to make sense of. However, there is at least one textbook series which includes lessons on how to draw and interpret double number lines explicitly (Tokyo Shoseki, 4B, pp. 132 - 133). I am hopeful that this will encourage more teachers to utilize double number lines in their lessons.

In order for students to use double number lines as a tool to determine the appropriate operation, students must be able to draw and interpret them. To nurture such an ability, I believe we need to start incorporating them in Grade 3 discussion of multiplication and division with whole numbers. Some textbooks do include some experiences that are foundational for double number lines in their units on whole number multiplication and division.

For example, here is an example from a Grade 3 textbook.

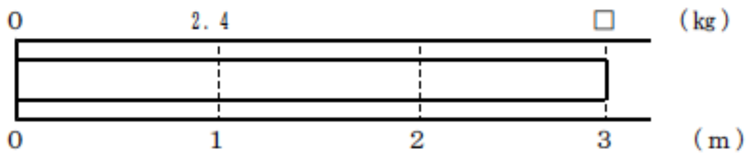


This diagram combines a tape diagram with a numberline. By learning to read this diagram carefully, students can nurture the ability to interpret double number lines.

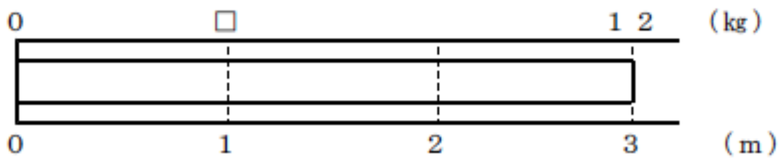
Students need to think about what "1" on the number line and "5" directly above it represent. They must also think about the meaning of "10" and "50," as well as the meaning of "20" and "100." We need to help students grasp that if the number of benches become 10 times as many, then the number of people who can be seated also becomes 10 times - and if the number of benches become 20 times, then the number of people also becomes 20 times. In other words, students grasp that the number of benches and the number of people are in proportion. Although they may hot know the term "proportion," they have a rudimentally concept of proportional relationships to make sense of the relationship.

If students can look at this diagram and grasp the relationshp among quantities, they can come up with the expression 5 × 30, and that means 30 times as many as 5 people. By accumulating this type of experiences, I believe students can gradually build their ability to interpret double number lines.

I want to propose to include diagrams like the ones shown below in the Grade 4 unit on multiplication and division of decimal numbers. For example, suppose the problem is: "1meter of iron bar weighs 2.4 kg. How much will 3 meters of the same iron bar weigh?" We can then show this diagram:



Or, if the problem is, "There is a 3-meter iron bar that weighs 12 kg. How much will 1 meter of the same iron bar weigh?" then the diagram will be like this:



Students may have difficulty interpreting double number lines since the two number lines show the relationship of quantities with numbers only. In the diagrams above, there is a tape diagram in between the two number lines, thus representing the problem situation a little more concretely. Perhaps students will have easier time interpreting these diagrams. By introducing diagrams like these before moving on to double number lines, we may be able to scaffold students' learning. Some textbook series began to include this type of diagram in the most recent edition.